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Higher-order conditions for singular Lagrangian dynamics

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Abstract. For a kth-order singular Lagrangian the presymplectic equation plus the different 'mth-order differential equation' conditions that can be considered in the Lagrangian space yield different dynamics which are studied. This leads to a new classification of the constraints. This study is also performed in the 'intermediate formalisms' which can be defined between the Lagrangian and the Hamiltonian ones; the corresponding classification schemes are then related.

1. Introduction

This paper deals with the dynamics constructed from a higher-order singular Lagrangian function. In order to make the ideas clear, let us first explain the relevant features of the first-order Lagrangian case.

For a first-order singular Lagrangian L the Euler–Lagrange equations of motion for a vector field X in the velocity space $\mathrm{T}(Q)$ can be written as the couple of equations†

$$\mathrm{i}_X \omega_L \simeq \mathrm{d} E_L \tag{1.1}$$

(the presymplectic equation) and

$$\mathrm{T}(o_Q) \circ X \simeq \mathrm{Id}_{\mathrm{T}(Q)} \tag{1.2}$$

(the second-order condition). Therefore one can apply the machinery of the presymplectic formalism and in particular its stabilization algorithm [11] to the Lagrangian formalism, introducing the second-order condition in a further step [10]. The first step yields a submanifold $P \subset \mathrm{T}(Q)$ where the presymplectic equation has solution; the second step results in a submanifold $S \subset P$ and a vector field tangent to S, which are a solution of the Euler–Lagrange equation. But such submanifold S is not maximal, and, in contrast with the presymplectic constraint algorithm, its construction from P is not algorithmic.

† The notation $\underset{S}{\simeq}$ means 'equality on the submanifold S' (Dirac's weak equality). When we state a property concerning a weak equality, it is understood to hold only at the points of S. Since S plays no role in what follows it will be suppressed.

An alternative approach which avoids this asymmetric situation can be followed through the concept of vector field along a map. There exists a vector field K along the Legendre's transformation $\mathrm{F}L$, such that the Euler–Lagrange equation is equivalently written [12] within a single equation for X:

$$\mathrm{T}(\mathrm{F}L) \circ X \simeq K. \tag{1.3}$$

Then a stabilization procedure can be performed, following the lines of [13, 14], in order to determine the (maximal) constraint submanifold and the dynamical fields of the Lagrangian formalism. Moreover, this procedure is strongly related to the Hamiltonian stabilization algorithm, since application of K (as a differential operator) to the Hamiltonian constraints yields the Lagrangian constraints [2, 19].

In spite of this, the consideration of the single presymplectic equation (1.1) has a certain interest. Indeed the constraints arising from it—the *presymplectic constraints*—obviously are constraints of the 'full dynamics' given by the equation of motion (1.3). In general, when the second-order condition is added to the presymplectic equation more constraints appear. Then, one can find a characterization of the presymplectic constraints: they are just the $\mathrm{F}L$-projectable constraints of the 'full dynamics' [18]; that is to say, they can be obtained as $\mathrm{F}L^*(\phi)$, where ϕ runs over the secondary Hamiltonian constraints. Therefore the 'non-presymplectic constraints' are not $\mathrm{F}L$-projectable; indeed they appear when some arbitrary functions of the Hamiltonian dynamics are determined through the stabilization algorithm [19].

Therefore the classification of the constraints upon their presymplectic or non-presymplectic 'origin' gives a deeper insight on the structure of Lagrangian constraints. Moreover, the knowledge of their projectability is useful when studying the existence of Noether gauge transformations [15].

When a kth-order singular Lagrangian is considered the situation is more involved because:

1. There appear $k-1$ intermediate spaces $P_0 \xrightarrow{\alpha_0} P_1 \rightarrow \ldots P_{k-1} \xrightarrow{\alpha_{k-1}} P_k$ between those of Lagrangian (P_0) and Hamiltonian (P_k) formalisms, where $\alpha_0, \ldots, \alpha_{k-1}$ are the 'partial Ostrogradskiĭ transformations' [3, 16]. All these spaces have their own dynamics, which are given by vector fields K_r along the maps α_r; if L is singular, there are also the corresponding constraints and arbitrary functions [16].
2. If L is singular, then $2(k-r)$ different dynamics can be constructed depending on the different 'higher-order conditions' to be considered in each space P_r, for $0 \leqslant r \leqslant k-1$.

In this paper we define and study these different dynamics and obtain some relations between the constraints depending upon their 'origin', that is to say, which mth-order condition is required to obtain them. Generalizing the first-order Lagrangian case we find a relation between the 'origin' of the constraints and their projectability through the partial Ostrogradskiĭ transformations; (it is known that this projectability is also related to the determination of arbitrary functions [16]).

To perform the analysis of the 'mth-order dynamics' we have also converted its two equations into a single one of type (1.3), which is treated as a 'linearly constrained dynamical system' of references [13, 14], whose machinery can be applied. These papers deal geometrically with a general type of singular differential equations, which includes all the constrained systems deduced from singular Lagrangians.

This work relies on our paper [16] where some geometric structures and dynamics for higher-order Lagrangians have been defined. We follow the notations and conventions in this paper. Some of our previous results and notation can be found in appendix A. On the other hand the proofs of section 3 are somewhat involved, so we have collected some auxiliary results in appendix B, in order to make the paper more readable.

We also assume a basic knowledge on vector bundles [1, 9], vector fields and sections along maps [4] (see also [6, 12]) and higher-order tangent bundles [20, 22] (see also [8, 17, 21], where higher-order Lagrangians are studied).

The paper is organized as follows. Section 2 contains the definition of the mth-order dynamics in the intermediate spaces. This dynamics is formulated into a single equation in section 3, from which some relations between the different constraints are obtained in section 4. In section 5 we discuss four examples which show different possibilities for the structure of constraints, and we compute which 'order condition' is needed to obtain them. Section 6 is devoted to conclusions, and finally there are two appendices with previous and auxiliary results.

2. Dynamics with the mth-order condition

Let us recall that our notations and previous results are explained in appendix A.

The equation of motion for a vector field X_r in P_r $(0 \leqslant r \leqslant k-1)$ is [16, theorem 6]

$$\mathrm{T}(\alpha_r) \circ X_r \simeq K_r. \tag{2.1}$$

This is an equation both for the submanifolds $S \subset P_r$ where the motion can take place and the vector fields X_r on P_r which are tangent to S.

The integral curves of X_r correspond to solutions of the Euler–Lagrange equation and therefore the coefficients of $\partial/\partial q^0, \ldots, \partial/\partial q^{2k-2-r}$ in X_r are perfectly determined: $q^1, \ldots, q^{2k-1-r}$. Thus, this vector field satisfies the $(2k-r)$th-order condition, which can be written

$$\mathrm{T}(o^{2k-1-r}_{2k-2-r}) \circ \mathrm{T}(\gamma_r) \circ X_r \simeq j^{2k-2-r} \circ \gamma_r. \tag{2.2}$$

Indeed, the equation of motion (2.1) is equivalent to this condition together with the presymplectic equation

$$\Omega_r \circ X_r \simeq \mathrm{d}E_r. \tag{2.3}$$

As done with first-order Lagrangians one may consider the presymplectic equation only and perform the presymplectic analysis on it [17]. For a regular Lagrangian the presymplectic equation already implies the $(2k-r)$th-order condition [16, proposition 7], but for a singular Lagrangian this result is no longer true. However:

Proposition 1. The presymplectic equation (2.3) in P_r $(0 \leqslant r \leqslant k-1)$ implies the $(r+1)$th-order condition for X_r.

Proof. This is a consequence of lemma 1 in appendix B. □

Thus we are faced, for $r+1 \leqslant m \leqslant 2k-r$, to the study of the *mth-order dynamics* in the intermediate space P_r: its solutions are the vector fields X_r in P_r satisfying the presymplectic equation (2.3) and the mth-order condition

$$\mathrm{T}(o^{2k-1-r}_{m-2}) \circ \mathrm{T}(\gamma_r) \circ X_r \simeq j^{m-2} \circ o^{2k-1-r}_{m-1} \circ \gamma_r . \tag{2.4}$$

In general, these $2(k-r)$ dynamics are not equivalent. The cases $m = 2k-r$, $r+1$ correspond to single equations (2.1) and (2.3), respectively.

In references [13, 14] a unified framework has been presented to deal with various constrained systems. We shall show in the following section that this framework also includes the mth-order dynamical formalisms just defined, which can be formulated within a unique equation instead of two.

Let us finally remark that for the Hamiltonian space P_k the maximal order that can be considered is kth, and the presymplectic equation on the primary constraint submanifold $P_k^{(0)}$ already implies it.

3. Equations of motion for the mth-order dynamics

Proposition 2. The equations of motion of the mth-order dynamics, for $2k-r \geqslant m \geqslant k+1$, are equivalent to

$$\mathrm{T}(\alpha_{r,2k+1-m}) \circ X_r \simeq K_{r,2k+1-m} . \tag{3.1}$$

Proof. The equation considered implies (2.3) and (2.4):

$$\begin{aligned}
\Omega_r \circ X_r &= {}^t\mathrm{T}(\alpha_{r,2k+1-m}) \circ \Omega_{2k+1-m} \circ \mathrm{T}(\alpha_{r,2k+1-m}) \circ X_r \\
&\simeq {}^t\mathrm{T}(\alpha_{r,2k+1-m}) \circ \Omega_{2k+1-m} \circ K_{r,2k+1-m} \\
&= {}^t\mathrm{T}(\alpha_{r,2k+1-m}) \circ \Omega_{2k+1-m} \circ \mathrm{T}(\alpha_{r+1,2k+1-m}) \circ K_r \\
&= {}^t\mathrm{T}(\alpha_r) \circ \Omega_{r+1} \circ K_r \\
&= \mathrm{d}E_r
\end{aligned}$$

where (B.2), (3.1), definition of K_{rs} (A.14) and proposition 7 (appendix B) have been used; and

$$\begin{aligned}
\mathrm{T}(o^{2k-1-r}_{m-2}) \circ \mathrm{T}(\gamma_r) \circ X_r &= \mathrm{T}(\gamma_{2k+1-m}) \circ \mathrm{T}(\alpha_{r,2k+1-m}) \circ X_r \\
&\simeq \mathrm{T}(\gamma_{2k+1-m}) \circ K_{r,2k+1-m} \\
&= j^{m-2} \circ \gamma_{2k-m} \circ \alpha_{r,2k-m} \\
&= j^{m-2} \circ o^{2k-1-r}_{m-1} \circ \gamma_r
\end{aligned}$$

where (B.1), (3.1) and proposition 7 have been used.

Conversely, if X_r satisfies (2.3) and (2.4) then (3.1) holds. To this end the characterization in proposition 7 is used to show that $\mathrm{T}(\alpha_{r,2k+1-m}) \circ X_r$ is $K_{r,2k+1-m}$ (on a certain submanifold):

$$
\begin{aligned}
{}^t\mathrm{T}(\alpha_{r,2k+1-m}) \circ \Omega_{2k+1-m} \circ (\mathrm{T}(\alpha_{r,2k+1-m}) \circ X_r) &= \Omega_r \circ X_r \\
&\simeq \mathrm{d}E_r
\end{aligned}
$$

$$
\begin{aligned}
\mathrm{T}(\gamma_{2k+1-m}) \circ (\mathrm{T}(\alpha_{r,2k+1-m}) \circ X_r) &= \mathrm{T}(o^{2k-1-r}_{m-2}) \circ \mathrm{T}(\gamma_r) \circ X_r \\
&\simeq j^{m-2} \circ o^{2k-1-r}_{m-1} \circ \gamma_r \\
&= j^{m-2} \circ \gamma_{2k-m} \circ \alpha_{r,2k-m}
\end{aligned}
$$

where we have used (B.2), (2.3), (B.1) and (2.4). □

Proposition 3. The equations of motion of the mth-order dynamics, for $k \geqslant m \geqslant r+1$, are equivalent to

$$
\Omega_{m-1} \circ \mathrm{T}(\alpha_{r,m-1}) \circ X_r \simeq \mathrm{d}E_{m-1} \circ \alpha_{r,m-1}. \tag{3.2}
$$

Proof. This equation implies (2.3):

$$
\begin{aligned}
\Omega_r \circ X_r &= {}^t\mathrm{T}(\alpha_{r,m-1}) \circ \Omega_{m-1} \circ \mathrm{T}(\alpha_{r,m-1}) \circ X_r \\
&\simeq {}^t\mathrm{T}(\alpha_{r,m-1}) \circ \mathrm{d}E_{m-1} \circ \alpha_{r,m-1} \\
&= \mathrm{d}E_r
\end{aligned}
$$

where (B.2), (3.2) and (B.3) have been used. On the other hand, by lemma 1, $\mathrm{T}(\alpha_{r,m-1}) \circ X_r$ satisfies the mth-order condition. Then, by lemma 2, X_r also satisfies the mth-order condition.

Conversely, let us deduce relation (3.2) from the equations of motion of the mth-order dynamics. First notice that, for $m = r+1$, (3.2) is nothing but the presymplectic equation. Therefore only the case $m \geqslant r+2$ is to be considered. From the presymplectic equation one has, using (B.2) and (B.3),

$$
{}^t\mathrm{T}(\alpha_{r,m-1}) \circ \left(\Omega_{m-1} \circ \mathrm{T}(\alpha_{r,m-1}) \circ X_r - \mathrm{d}E_{m-1} \circ \alpha_{r,m-1}\right) \simeq 0.
$$

Since X_r satisfies the mth-order condition, $\mathrm{T}(\alpha_{r,m-1}) \circ X_r$ also satisfies it. By lemma 4 the expression in parentheses is zero, and this is equation (3.2). □

Theorem 1. For each intermediate space P_r $(0 \leqslant r \leqslant k-1)$ and integer m $(r+1 \leqslant m \leqslant 2k-r)$ there exist a vector bundle $\pi_{rm}\colon F_{rm} \to P_r$, a morphism of vector P_r-bundles $A_{rm}\colon \mathrm{T}(P_r) \to F_{rm}$, and a section σ_{rm} of F_{rm}, such that the equations of motion (2.3) and (2.4) for a vector field with the mth-order dynamics in P_r are equivalent to the single equation

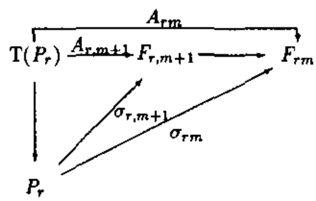
$$
A_{rm} \circ X_r \simeq \sigma_{rm}. \tag{3.3}
$$

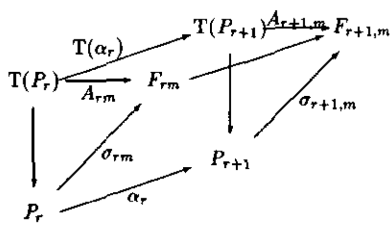
The same is true for the Hamiltonian formalism considering $m = k$ and the corresponding vector bundles on the base $P_k^{(1)} = \alpha_{k-1}(P_{k-1})$, provided it is a submanifold of P_k and the Hamiltonian formalism can be defined on it.

The various dynamics of P_r are naturally related by P_r-morphisms $F_{r,m+1} \longrightarrow F_{rm}$ $(r+1 \leqslant m \leqslant 2k-1-r)$.

The various dynamics with the mth-order condition are naturally related by P_r-*iso*morphisms $F_{rm} \to P_r \times_{\alpha_r} F_{r+1,m}$ (where $0 \leqslant r \leqslant \min\{m-2, 2k-1-m\}$ if $m \neq k$, $0 \leqslant r \leqslant k-1$ if $m = k$).

These morphisms yield the following commutative diagrams:





Proof. If $2k - r \geqslant m \geqslant k+1$ put

$$F_{rm} = P_r \times_{\alpha_{r,2k+1-m}} \mathrm{T}(P_{2k+1-m}) \tag{3.4a}$$

$$A_{rm} = \mathrm{T}(\alpha_{r,2k+1-m}) \tag{3.4b}$$

$$\sigma_{rm} = K_{r,2k+1-m} \tag{3.4c}$$

and if $k \geqslant m \geqslant r+1$ put

$$F_{rm} = P_r \times_{\alpha_{r,m-1}} \mathrm{T}(P_{m-1})^* \tag{3.5a}$$

$$\begin{aligned} A_{rm} &= \Omega_{m-1} \circ \mathrm{T}(\alpha_{r,m-1}) \\ &= {}^t\mathrm{T}(\alpha_{m-1,k}) \circ \Omega_k \circ \mathrm{T}(\alpha_{r,k}) \end{aligned} \tag{3.5b}$$

$$\begin{aligned} \sigma_{rm} &= \mathrm{d}E_{m-1} \circ \alpha_{r,m-1} \\ &\underset{m>r+1}{=} {}^t\mathrm{T}(\alpha_{m-1,k}) \circ \Omega_k \circ K_{rk} \end{aligned} \tag{3.5c}$$

where A_{rm} is regarded as a morphism of vector P_r-bundles $\mathrm{T}(P_r) \to F_{rm}$ and σ_{rm} as a section of F_{rm}. Thus, according to the preceding propositions, the equations of motion for the mth-order dynamics are equivalent to (3.3). It is clear that the various dynamics in P_r are related by morphisms between the vector P_r-bundles $\mathrm{T}(P_r)$, $F_{r,2k-r}, \ldots, F_{r,r+1}$:

$$\begin{aligned} &\mathrm{T}(P_r) \overset{\mathrm{T}(\alpha_r)}{\to} P_r \times_{\alpha_r} \mathrm{T}(P_{r+1}) \overset{\mathrm{T}(\alpha_{r+1})}{\to} \ldots \to P_r \times_{\alpha_{rk}} \mathrm{T}(P_k) \\ &\quad \overset{\Omega_k}{\cong} P_r \times_{\alpha_{rk}} \mathrm{T}(P_k)^* \to \ldots \overset{{}^t\mathrm{T}(\alpha_{r+1})}{\to} P_r \times_{\alpha_r} \mathrm{T}(P_{r+1})^* \overset{{}^t\mathrm{T}(\alpha_r)}{\to} \mathrm{T}(P_r)^* \end{aligned}$$

since their compositions yield the A_{rm} and they transform the corresponding sections σ_{rm}.

On the other hand, since the vector bundle F_{rm} is the inverse image by partial Ostrogadskiĭ transformations of a vector bundle depending on m and not on r, and the spaces P_r are related through the partial Ostrogadskiĭ transformations, there are natural isomorphisms $F_{rm} \to P_r \times_{\alpha_r} F_{r+1,m}$.

The particular statements concerning the Hamiltonian formalism (where $r = m = k$) can be easily checked. □

The data $(P_r, F_{rm}, \pi_{rm}, A_{rm}, \sigma_{rm})$ constitute a *linearly constrained system* [13, 14], whose equation of motion (for a vector field) is (3.3).

4. Relations between the constraints

The morphisms relating the various dynamics imply the following results (see also [13, 14]). Let $S_{rm} \subset P_r$ be the *final constraint submanifold* of the mth-order dynamics in P_r; that is to say, the subset of P_r, *assumed to be a closed submanifold*, of points by which there pass solutions of the equation of motion. Let $C_{rm} \subset \mathrm{C}^\infty(P_r)$ be the ideal of functions vanishing on S_{rm}. Then:

Proposition 4. If ξ_r is a path in P_r solution of the $(m+1)$th-order dynamics, then it is a solution of the mth-order dynamics. Therefore, $S_{r,m+1} \subset S_{rm}$ and $C_{rm} \subset C_{r,m+1}$.

This result was expected since the $(m+1)$th-order dynamics is a restriction of that of mth-order. The proof is similar but simpler than that of the following proposition.

Now let $C_{rm}^{\mathrm{proj}} := C_{rm} \cap \alpha_r^*(\mathrm{C}^\infty(P_{r+1}))$ be the subset of α_r-projectable constraints of C_{rm}.

Proposition 5. Let ξ_r be a path in P_r, and let $\xi_{r+1} = \alpha_r \circ \xi_r$ be the corresponding path in P_{r+1}. Then ξ_r is a solution of the mth-order dynamics if and only if ξ_{r+1} is. Therefore, $\alpha_r(S_{rm}) \subset S_{r+1,m}$ and $\alpha_r^*(C_{r+1,m}) \subset C_{rm}^{\mathrm{proj}}$.

Proof. Suppose that ξ_r is a solution. With the notations in the theorem, its second commutative diagram yields

$$\begin{aligned} A_{r+1,m} \circ \dot{\xi}_{r+1} &= A_{r+1,m} \circ \mathrm{T}(\alpha_r) \circ \dot{\xi}_r \\ &= \sigma_{r+1,m} \circ \alpha_r \circ \xi_r \\ &= \sigma_{r+1,m} \circ \xi_{r+1}. \end{aligned}$$

Similarly one shows that if ξ_{r+1} is a solution then ξ_r is, using the linear isomorphisms $F_{rm,x} \to F_{r+1,m,x}$.

Then the relations between the final constraint submanifolds and the ideals of constraints are easily deduced. □

In fact the relations between the constraint submanifolds *are true at each level of the stabilization algorithms* of these dynamics [14].

The following diagram shows all the relations established between the constraints:

$$
\begin{array}{ccccccc}
C_{r,r+1} & \supset & C^{\rm proj}_{r,r+1} & & & & \\
\cap & & \cap & & & & \\
C_{r,r+2} & \supset & C^{\rm proj}_{r,r+2} & \supset & \alpha_r^*(C_{r+1,r+2}) & \stackrel{\alpha_r^*}{\longleftarrow} & C_{r+1,r+2} \\
\cap & & \cap & & \cap & & \cap \\
\vdots & & \vdots & & \vdots & & \vdots \\
C_{r,m} & \supset & C^{\rm proj}_{r,m} & \supset & \alpha_r^*(C_{r+1,m}) & \stackrel{\alpha_r^*}{\longleftarrow} & C_{r+1,m} \\
\vdots & & \vdots & & \vdots & & \vdots \\
\cap & & \cap & & \cap & & \cap \\
C_{r,2k-r-1} & \supset & C^{\rm proj}_{r,2k-r-1} & \supset & \alpha_r^*(C_{r+1,2k-r-1}) & \stackrel{\alpha_r^*}{\longleftarrow} & C_{r+1,2k-r-1} \\
\cap & & & & & & \\
C_{r,2k-r} & & & & & &
\end{array}
$$

Proposition 5 can be improved under additional hypotheses:

Proposition 6. Assume moreover that $S_{r+1,m} \subset \alpha_r(P_r) \subset P_{r+1}$ are closed submanifolds. Then $\alpha_r^{-1}(S_{r+1,m}) = S_{rm}$, $\alpha_r(S_{rm}) = S_{r+1,m}$ and C_{rm} is the ideal generated by $\alpha_r^*(C_{r+1,m})$.

Proof. The inclusion $\alpha_r^{-1}(S_{r+1,m}) \supset S_{rm}$ is already known. Now let $x \in \alpha_r^{-1}(S_{r+1,m})$, and ξ_{r+1} a solution passing by $x' = \alpha_r(x)$. There is a path ξ_r passing by x such that $\xi_{r+1} = \alpha_r \circ \xi_r$. By proposition 5, ξ_r is a solution of the mth-order dynamics, thus $x \in S_{rm}$. The same argument proves the second equality. Finally, the relation between the constraints follows from the relation between the submanifolds. □

Therefore, in the row corresponding to the mth-order dynamics in the preceding diagram the horizontal inclusions are equalities. Let us call *presymplectic constraints* of P_r the constraints arising from the presymplectic equation; therefore, those in $C_{r,r+1}$. Using proposition 4 we conclude that:

Corollary. If $S_{r+1,r+2} \subset \alpha_r(P_r)$ (in other words, the constraints defining $\alpha_r(P_r) \subset P_{r+1}$ are presymplectic constraints) then the conclusions of proposition 6 hold for each m. In particular, the pull-back through α_r of all the constraints in P_{r+1} $(C_{r+1,2k-r-1})$ generates $C_{r,2k-r-1}$; and the pull-back through α_r of the presymplectic constraints in P_{r+1} $(C_{r+1,r+2})$ generates $C_{r,r+2}$, thus it may include presymplectic and non-presymplectic constraints.

Now all the horizontal inclusions in the preceding diagram are equalities.

Notice that the hypothesis in the corollary does not hold in general. The primary constraints appearing by the compatibility condition of the equation of motion (2.1) are those defining the submanifold $\alpha_r(P_{r-1}) \subset P_r$ [16, proposition 9]. On the other hand, using that $\mathrm{Ker}\,\mathrm{T}(\alpha_r) \subset \mathrm{Ker}\,\Omega_r$, it can be shown that the primary constraints appearing by the compatibility condition of the presymplectic equation (2.3) are $\chi_r^{\mu} := \tilde{\Gamma}^r_{\mu'} \cdot E_r$, where the vector fields $\tilde{\Gamma}^r_{\mu'}$ constitute, in addition to a frame of $\mathrm{Ker}\,\mathrm{T}(\alpha_r)$, a frame for $\mathrm{Ker}\,\Omega_r$.

5. Examples

Here we consider four second-order singular Lagrangians and perform the constraint analysis in the spaces $P_0 \cong \mathrm{T}^3Q$ (Lagrangian formalism), P_1 (intermediate formalism), and $P_2 \cong \mathrm{T}(\mathrm{T}Q)^*$ (Hamiltonian formalism), considering also the different order conditions that can be imposed on each space. The first example is done in more detail. The procedure followed is similar to that of [18] for first-order Lagrangians.

Of course, if only the maximal-order condition is to be considered then the results of [16] can be applied. Indeed, it is easily checked that if the Dirac Hamiltonian analysis is first performed and the intermediate evolution operators K_1 and K_0 are applied to the Hamiltonian constraints, then all the constraints (of the maximal-order dynamics) are obtained as given in the examples.

We want to point out that all the examples satisfy the conclusion of the corollary to proposition 6 for $r = 0$, but the third one does not satisfy its hypothesis. We do not know whether this hypothesis can be weakened.

Example 1. $L(x^0, x^1, x^2) = \frac{1}{2}x^1x^1$.

(a) Lagrangian formalism (in P_0). We need $\omega_0 = \mathrm{d}x^0 \wedge \mathrm{d}x^1$ and $E_0 = \frac{1}{2}x^1x^1$. Let† $X_0 = A^0\partial_0 + \ldots + A^3\partial_3$ be the dynamical field, with $A^i(x^0, \ldots, x^3)$ functions. The primary constraints are obtained by requiring $\mathrm{i}_{X_0}\omega_0 \simeq \mathrm{d}E_0$, i.e., $A^0\mathrm{d}x^1 - A^1\mathrm{d}x^0 \simeq x^1\mathrm{d}x^1$.

(i) Presymplectic equation. It determines $A^0 = x^1$ and $A^1 = 0$, therefore there are no constraints and the dynamical field is $X_0 = x^1\partial_0 + A^2\partial_2 + A^3\partial_3$.

(ii) Second-order dynamics. We previously set $A^0 = x^1$, but this is already satisfied by the solutions of the presymplectic equation. Therefore there are no constraints and still $X_0 = x^1\partial_0 + A^2\partial_2 + A^3\partial_3$.

(iii) Third-order dynamics. Now the dynamical field is looked for in the form $X_0 = x^1\partial_0 + x^2\partial_1 + A^2\partial_2 + A^3\partial_3$. Equation $\mathrm{i}_{X_0}\omega_0 \simeq \mathrm{d}E_0$ implies that $x^2 \simeq 0$. In other words, $\phi_0^1 = x^2$ is the primary constraint and the dynamical field is $X_0 \simeq x^1\partial_0 + A^2\partial_2 + A^3\partial_3$. The stability of this constraint imposes $X_0 \cdot \phi_0^1 \simeq A^2 \simeq 0$, which determines $X_0 \simeq x^1\partial_0 + A^3\partial_3$ and no more constraints.

(iv) Full dynamics. We first set $X_0 = x^1\partial_0 + \ldots + x^3\partial_2 + A^3\partial_3$. Equation $\mathrm{i}_{X_0}\omega_0 \simeq \mathrm{d}E_0$ produces the primary constraint $\phi_0^1 = x^2$. Its stability is the secondary constraint $\phi_0^2 = X_0 \cdot \phi_0^1 = x^3$, and the stability of ϕ_0^2 only determines $X_0 \cdot \phi_0^2 \simeq A^3 \simeq 0$. Therefore no more constraints appear and the dynamical field is $X_0 \simeq x^1\partial_0$.

(b) Intermediate formalism (in P_1). The partial Ostrogadskiĭ transformation $\alpha_0 \colon P_0 \to P_1$ is defined by $\hat{p}_0 = x^1$ (which will be the primary constraint of the full dynamics), and $\omega_1 = \mathrm{d}x^0 \wedge \mathrm{d}p_0$ and $E_1 = (p_0 - \frac{1}{2}x^1)x^1$.

Let $X_1 = A^0\partial_0 + A^1\partial_1 + A^2\partial_2 + C_0\partial^0$ be the dynamical field. The primary constraints are obtained by requiring $i_{X_1}\omega_1 \simeq \mathrm{d}E_1$, i.e., $A_0\mathrm{d}p_0 - C_0\mathrm{d}x^0 \simeq x^1\mathrm{d}p_0 + (p_0 - x^1)\mathrm{d}x^1$.

(i) Presymplectic equation. It is known that $A_0 \simeq x^1$. Moreover, $C_0 = 0$ and $\phi_1^1 = x^1 - p_0$ is the primary constraint. Now $X_1 \simeq x^1\partial_0 + A^1\partial_1 + A^2\partial_2$. Imposing the stability of ϕ_1^1 leads to $X_1 \cdot \phi_1^1 = A^1 \simeq 0$, which finally determines $X_1 \simeq x^1\partial_0 + A^2\partial_2$.

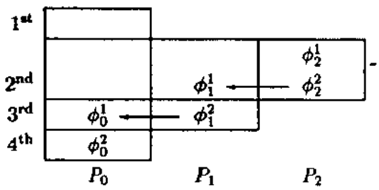
† We put $\partial_i = \partial/\partial x^i$ and $\partial^j = \partial/\partial p_j$.

(ii) Full dynamics. Now $X_1 = x^1\partial_0 + x^2\partial_1 + A^2\partial_2 + C_0\partial^0$, and also $C_0 = 0$ and $\phi_1^1 = x^1 - p_0 \simeq 0$. The stability of ϕ_1^1 is the secondary constraint $\phi_1^2 = X_1 \cdot \phi_1^1 = x^2$ whose stability determines $A^2 \simeq 0$ and the dynamical field $X_1 \simeq x^1\partial_0$.

(c) Hamiltonian formalism (in P_2). The partial Ostrogadskiĭ transformation $\alpha_1 : P_1 \to P_2$ is defined by $\hat{p}_1 = 0$, which gives the primary Hamiltonian constraint $\phi_2^1 = p_1$. Now $E_2 = H \simeq (p_0 - \frac{1}{2}x^1)x^1$, which gives a secondary constraint $\phi_2^2 = x^1 - p_0$, already stable. The dynamics is given by $X_2 = x^1\partial_0$.

Notice that the Hamiltonian constraints are second class. Since one of them is primary, the secondary constraint ϕ_1^2 of P_1 is not α_1-projectable [16], and therefore it was expected to be non-presymplectic. The same applies to ϕ_0^2. On the other hand, as said before, we have $K_1 \cdot \phi_2^i = \phi_1^i$, and $K_0 \cdot \phi_1^i = \phi_0^i$; this also holds in the following examples.

To summarize, we have the following scheme, where the arrows denote pull-back through partial Ostrogadskiĭ transformations:



In the following examples we only give the constraints of the full—maximal-order condition—dynamics, and their classification like in the preceding example.

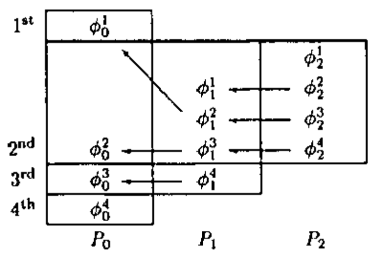
Example 2. $L(x^0, x^1, x^2, y^0, y^1, y^2) = \frac{1}{2}x^2x^2 + x^2y^0$ [5].

(a) Lagrangian formalism: We now look for a dynamical field $X_0 = A^0\partial_{x0} + \ldots + A^3\partial_{x3} + B^0\partial_{y0} + \ldots + B^3\partial_{y3}$. The constraints are $\phi_0^1 = -x^2$, $\phi_0^2 = -x^3$, $\phi_0^3 = y^2$ and $\phi_0^4 = y^3$.

(b) Intermediate formalism: $\phi_1^1 = -p_{y0}$, $\phi_1^2 = -x^2$, $\phi_1^3 = y^1 + p_{x0}$ and $\phi_1^4 = y^2$.

(c) Hamiltonian formalism: $\phi_2^1 = p_{y1}$, $\phi_2^2 = -p_{y0}$, $\phi_2^3 = y^0 - p_{x1}$ and $\phi_2^4 = y^1 + p_{x0}$.

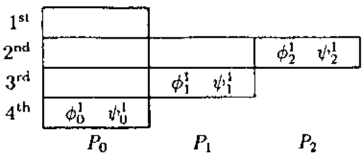
Here all the Hamiltonian constraints are also second class, and then one 'maximal-order' constraint in P_1 and P_0 is expected. The result is:



Example 3. $L(x^0, x^1, x^2, y^0, y^1, y^2) = x^1 y^2$.

(a) Lagrangian formalism: $\phi_0^1 = x^3$ and $\psi_0^1 = y^3$.
(b) Intermediate formalism: $\phi_1^1 = x^2 + p_{y0}$ and $\psi_1^1 = y^2 - p_{x0}$.
(c) Hamiltonian formalism: $\phi_2^1 = x^1 - p_{y1}$ and $\psi_2^1 = p_{x1}$.

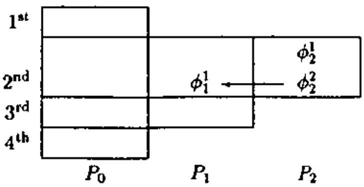
As said before one finds here that the presymplectic equation in P_0 and P_1 does not produce any constraint. Both primary Hamiltonian constraints are second class; therefore projectable constraints are not expected. This is true since the constraints of P_0 and P_1 arise only from the maximal-order condition:



Example 4. $L(x^0, x^1, x^2) = x^2$.

(a) Lagrangian formalism: There are no Lagrangian constraints.
(b) Intermediate formalism: The only constraint is $\phi_1^1 = p_0$.
(c) Hamiltonian formalism: $\phi_2^1 = 1 - p_1$ and $\phi_2^2 = p_0$.

Since the two Hamiltonian constraints are first class the secondary one gives through α_1^* the only constraint in P_1, which is presymplectic:



6. Conclusions

We have studied the dynamics in the Lagrangian, Hamiltonian and, in general, intermediate formalisms in the framework introduced in [16] for higher-order Lagrangians. These dynamics are constructed depending on which 'mth-order differential equation' condition is required; for singular Lagrangians there appear different dynamics and a new classification scheme of the constraints is thus obtained.

We have shown that all the dynamics herein defined have an intrinsic formulation by means of a single equation. This approach eases to relate the classification of the constraints with their projectability through the partial Ostrogradskiĭ transformations. This is a clear way to show how the constraints appear in the formalism according to some specific dynamical requirements. The best result is obtained under some hypotheses (proposition 6); then the pull-backs (through partial Ostrogradskiĭ transformations) of the mth-order constraints are just the mth-order constraints, and the non-projectable constraints are just those arising from the highest-order condition that can be considered.

These refinement in the classification of constraints in higher-order theories could be useful—as they are in the first-order case—in the obtention of Lagrangian gauge transformations.

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Appendix A. Previous results

Here we present some previous results and notation from reference [16].

We consider an n-dimensional differentiable manifold Q with coordinates† $q = q^0$, and its higher-order tangent bundles $\mathrm{T}^r Q$, with natural coordinates $(q^0, \ldots, q^r)$ A kth-order Lagrangian is a function $L \in \mathrm{C}^\infty(\mathrm{T}^k Q)$. The Jacobi–Ostrogradskiĭ momenta are

$$\hat{p}_i = \sum_{j=0}^{k-i-1} (-1)^j \mathrm{d}_\mathrm{T}^j \left(\frac{\partial L}{\partial q^{i+j+1}} \right) \qquad 0 \leqslant i \leqslant k-1 \tag{A.1}$$

and satisfy

$$\hat{p}_{i-1} = \frac{\partial L}{\partial q^i} - \mathrm{d}_\mathrm{T} \hat{p}_i \tag{A.2}$$

where $\mathrm{d}_\mathrm{T} = \sum_i q^{i+1} \partial / \partial q^i$.

The intermediate space P_r $(0 \leqslant r \leqslant k)$ can be defined, with coordinates $(q^0, \ldots, q^{2k-1-r}; p_0, \ldots, p_{r-1})$. Then $P_0 \simeq \mathrm{T}^{2k-1}Q$ and $P_k \simeq \mathrm{T}^*(\mathrm{T}^{k-1}Q)$. We set $\gamma_r \colon P_r \to \mathrm{T}^{2k-1-r}Q$ for the natural projection.

A tangent vector $v_x \in \mathrm{T}_x(P_r)$ satisfies the *mth-order condition* if its projection to $\mathrm{T}^{2k-1-r}Q$ satisfies it. That is to say,

$$\mathrm{T}(o^{2k-1-r}_{m-2}) \circ \mathrm{T}(\gamma_r) \cdot v_x = j^{m-2} \circ o^{2k-1-r}_{m-1} \circ \gamma_r(x) \tag{A.3}$$

where $o^s_l \colon \mathrm{T}^s Q \to \mathrm{T}^l Q$ is the canonical projection and $j^s \colon \mathrm{T}^{s+1} Q \to \mathrm{T}(\mathrm{T}^s Q)$ is the canonical embedding.

The partial Ostrogradskiĭ transformations $\alpha_r \colon P_r \to P_{r+1}$ can be introduced, with local expression

$$\alpha_r(q^0, \ldots, q^{2k-1-r}; p_0, \ldots, p_{r-1}) = (q^0, \ldots, q^{2k-2-r}; p_0, \ldots, p_{r-1}, \hat{p}_r). \tag{A.4}$$

For $0 \leqslant r \leqslant s \leqslant k$ we also use $\alpha_{rs} \colon P_r \to P_s$, defined as the composition

$$\alpha_{rs} := \alpha_{s-1} \circ \ldots \circ \alpha_r. \tag{A.5}$$

The 'total' Legendre–Ostrogradskiĭ transformation is therefore $\mathrm{F}L = \alpha_{0k} \colon P_0 \to P_k$.

† Indices of coordinates are always suppressed.

With α_{rk} we construct the exact 2-form $\omega_r := \alpha_{rk}^*(\omega)$, where $\omega = \omega_k$ is the canonical symplectic form of $\mathrm{T}^*(\mathrm{T}^{k-1}Q)$. Its local expression is

$$\omega_r = \mathrm{d}q^0 \wedge \mathrm{d}p_0 + \ldots + \mathrm{d}q^{r-1} \wedge \mathrm{d}p_{r-1} + \mathrm{d}q^r \wedge \mathrm{d}\hat{p}_r + \ldots + \mathrm{d}q^{k-1} \wedge \mathrm{d}\hat{p}_{k-1} \quad \text{(A.6)}$$

and by inner contraction it defines a morphism of vector bundles

$$\Omega_r : \mathrm{T}(P_r) \to \mathrm{T}(P_r)^*. \quad \text{(A.7)}$$

On the other hand, in P_r $(0 \leqslant r \leqslant k-1)$ there is the energy function E_r, locally given by

$$E_r(q^0, \ldots, p_{r-1}) = p_0 q^1 + \ldots + p_{r-1} q^r + \hat{p}_r q^{r+1} + \ldots + \hat{p}_{k-1} q^k - L(q^0, \ldots, q^k). \quad \text{(A.8)}$$

We have $E_r = \alpha_r^*(E_{r+1})$.

Now we consider the intermediate evolution operator K_r, which is the only vector field along α_r satisfying the following two conditions [16, theorem 4]:

$$\mathrm{T}(\gamma_{r+1}) \circ K_r = j^{2k-2-r} \circ \gamma_r \quad \text{(A.9)}$$

$$\alpha_r^* \left(\mathrm{i}_{K_r} \omega_{r+1}\right) = {}^t\mathrm{T}(\alpha_r) \circ \Omega_{r+1} \circ K_r = \mathrm{d}E_r. \quad \text{(A.10)}$$

In coordinates it reads

$$K_r = q^1 \frac{\partial}{\partial q^0} + \ldots + q^{2k-1-r} \frac{\partial}{\partial q^{2k-2-r}} + \\ + \left(\frac{\partial L}{\partial q^0}\right) \frac{\partial}{\partial p_0} + \left(\frac{\partial L}{\partial q^1} - p_0\right) \frac{\partial}{\partial p_1} + \ldots + \left(\frac{\partial L}{\partial q^r} - p_{r-1}\right) \frac{\partial}{\partial p_r}. \quad \text{(A.11)}$$

It is a differential operator on the functions in P_{r+1}:

$$K_r \cdot f := \langle \mathrm{d}f \circ \alpha_r, K_r \rangle. \quad \text{(A.12)}$$

The different evolution operators are connected by

$$\mathrm{T}(\alpha_r) \circ K_{r-1} = K_r \circ \alpha_{r-1}. \quad \text{(A.13)}$$

Then for $0 \leqslant r < s \leqslant k$ we define $K_{rs} : P_r \to \mathrm{T}(P_s)$ by

$$K_{rs} := K_{s-1} \circ \alpha_{r,s-1} = \mathrm{T}(\alpha_{r+1,s}) \circ K_r. \quad \text{(A.14)}$$

It is a vector field along α_{rs} and therefore it is identified with a section of $P_r \times_{\alpha_{rs}} \mathrm{T}(P_s)$.

It is assumed that the α_r have the same constant rank $2kn - m$ and that $P_{r+1}^{(1)} := \alpha_r(P_r)$ is a closed submanifold of P_{r+1} locally defined by m independent primary constraints ϕ_{r+1}^μ. The primary Hamiltonian constraints—those defining $P_k^{(1)}$—can be chosen to not depend on $p_0, \ldots, p_{k-2}$. Then the primary constraints of P_r can be obtained by applying K_r to the primary constraints of P_{r+1} [16, proposition 9].

There is a (local) Hamiltonian function in P_k, which can be chosen of the particular form

$$H = E_k = \sum_{i=0}^{k-2} p_i q^{i+1} + h(q^0, \ldots, q^{k-1}; p_{k-1}). \quad \text{(A.15)}$$

The usual presymplectic analysis can be performed in $P_k^{(1)}$. In fact, there are stabilization algorithms for the dynamics of the intermediate spaces given by equation (2.1), and *all* the constraints in P_r are obtained applying K_r to all the constraints in P_{r+1} [16, theorem 8]. Indeed, this result holds at each step of the stabilization algorithm.

Appendix B. Auxiliary results

First let us quote some relations that follow immediately from the preceding definitions.

From the commutation relation $o^{2k-1-r}_{2k-1-s} \circ \gamma_r = \gamma_{r+1} \circ \alpha_r$ one obtains, for $0 \leqslant r \leqslant s \leqslant k$,

$$o^{2k-1-r}_{2k-1-s} \circ \gamma_r = \gamma_s \circ \alpha_{rs}. \tag{B.1}$$

Using $\alpha_r^*(\omega_{r+1}) = \omega_r$ we have, for $0 \leqslant r \leqslant s \leqslant k$,

$$\Omega_r = {}^t\mathrm{T}(\alpha_{rs}) \circ \Omega_s \circ \mathrm{T}(\alpha_{rs}) \tag{B.2}$$

where $\mathrm{T}(\alpha_{rs})$, its transposed ${}^t\mathrm{T}(\alpha_{rs})$, and Ω_s are regarded as vector P_r-bundle morphisms. Similarly,

$$\mathrm{d}E_r = {}^t\mathrm{T}(\alpha_{rs}) \circ (\mathrm{d}E_s \circ \alpha_{rs}) \tag{B.3}$$

where $\mathrm{d}E_s \circ \alpha_{rs}$ is seen as a section of $P_r \times_{\alpha_{rs}} \mathrm{T}(P_s)^*$. Similar conventions will be used without further reference.

Lemma 1. Let $0 \leqslant r \leqslant k$, and a tangent vector $v_x \in \mathrm{T}_x(P_r)$. The local expression of $\Omega_r \cdot v_x - \mathrm{d}E_r(x)$ does not contain terms in the $\mathrm{d}p_j$ $(0 \leqslant j \leqslant r-1)$ if and only if v_x satisfies the condition of order $r+1$ if $r < k$, of order k if $r = k$.

Proof. From the local expression of ω_r (A.6) it is clear that if

$$v_x = \sum_{i=0}^{2k-1-r} v^i \frac{\partial}{\partial q^i} + \sum_{j=0}^{r-1} w_j \frac{\partial}{\partial p_j}$$

then

$$\Omega_r \cdot v_x = v^0 \mathrm{d}p_0 + \ldots + v^{r-1} \mathrm{d}p_{r-1} + \text{terms in the } \mathrm{d}q^i.$$

Then just compute the local expression of $\mathrm{d}E_r$ according to (A.8) and (A.15). □

Lemma 2. Let $0 \leqslant r \leqslant s \leqslant k$ and $2 \leqslant m \leqslant 2k - s$. A tangent vector $v_x \in \mathrm{T}_x(P_r)$ satisfies the mth-order condition if and only if $\mathrm{T}(\alpha_{rs}) \cdot v_x \in \mathrm{T}_{\alpha_{rs}(x)}(P_s)$ satisfies it also.

Proof. The condition $\mathrm{T}(o^{2k-1-r}_{m-2}) \circ \mathrm{T}(\gamma_r) \cdot v_x = j^{m-2} \circ o^{2k-1-r}_{m-1} \circ \gamma_r(x)$ is seen to be equivalent to $\mathrm{T}(o^{2k-1-s}_{m-2}) \circ \mathrm{T}(\gamma_s) \cdot (\mathrm{T}(\alpha_{rs}) \cdot v_x) = j^{m-2} \circ o^{2k-1-s}_{m-1} \circ \gamma_s(\alpha_{rs}(x))$ by decomposition of the projections from $\mathrm{T}^{2k-1-r}Q$ and use of (B.1). □

Lemma 3. Let $0 \leqslant r \leqslant s \leqslant k$. An element of $\mathrm{Ker}\,{}^t\mathrm{T}(\alpha_{rs})$ whose local expression does not contain terms in $\mathrm{d}p_j$ for $0 \leqslant j \leqslant s-1$ is zero.

Proof. The form of such an element is therefore $\sum_{i=0}^{2k-1-s} a_i \mathrm{d}q^i$. But the coordinates q^i for $0 \leqslant i \leqslant 2k-1-s$ are invariant through α_{rs}, thus ${}^t\mathrm{T}(\alpha_{rs})(\sum a_i \mathrm{d}q^i) = \sum a_i \mathrm{d}q^i = 0$, therefore the coefficents a_i are zero. □

As a corollary of this and lemma 1 we have:

Lemma 4. Let $0 \leqslant r \leqslant s \leqslant k$. If $v_x \in \mathrm{T}_x(P_s)$ satisfies the condition of order $s+1$ if $s < k$, of order k if $s = k$, and moreover ${}^t\mathrm{T}(\alpha_{rs}) \cdot (\Omega_s \cdot v_x - \mathrm{d}E_s(x)) = 0$, then $\Omega_s \cdot v_x = \mathrm{d}E_s(x)$.

Finally, some results on the evolution operators will be needed. It is known that ${}^t\mathrm{T}(\alpha_r) \circ \Omega_{r+1} \circ K_r = \mathrm{d}E_r = {}^t\mathrm{T}(\alpha_r) \circ \mathrm{d}E_{r+1} \circ \alpha_r$ for $0 \leqslant r < k$. Indeed, a careful observation of the corresponding local expressions shows that $\Omega_{r+1} \circ K_r = \mathrm{d}E_{r+1} \circ \alpha_r$ if $r < k-1$. This proves, more generally, for $0 \leqslant r < s < k$,

$$\Omega_s \circ K_{rs} = \mathrm{d}E_s \circ \alpha_{rs} . \tag{B.4}$$

The result analogous to [16, theorem 4] for K_{rs} is the following:

Proposition 7. (Characterization of K_{rs}). For $0 \leqslant r < s \leqslant k$, K_{rs} is the only vector field along α_{rs} such that satisfies the 'presymplectic condition'

$${}^t\mathrm{T}(\alpha_{rs}) \circ \Omega_s \circ K_{rs} = \mathrm{d}E_r \tag{B.5}$$

and the '$(2k+1-s)$th-order condition'

$$\mathrm{T}(\gamma_s) \circ K_{rs} = j^{2k-1-s} \circ \gamma_{s-1} \circ \alpha_{r,s-1} . \tag{B.6}$$

Proof. These conditions are clearly consequence of those satisfied by K_{s-1}. Conversely, let us rewrite the presymplectic condition as ${}^t\mathrm{T}(\alpha_{rs}) \circ (\Omega_s \circ K_{rs} - \mathrm{d}E_s \circ \alpha_{rs}) = 0$. Since the vectors image of K_{rs} satisfy the condition of order $2k-s$, which is $\geqslant s+1$ if $s < k$, and equal to k if $s = k$, it follows from lemma 4 that $\Omega_s \circ K_{rs} = \mathrm{d}E_s \circ \alpha_{rs}$. By applying ${}^t\mathrm{T}(\alpha_{s-1})$ we obtain ${}^t\mathrm{T}(\alpha_{s-1}) \circ \Omega_s \circ K_{rs} = \mathrm{d}E_{s-1} \circ \alpha_{r,s-1}$, which, together with the $(2k+1-s)$th-order condition, is the characterization of $K_{rs} = K_{s-1} \circ \alpha_{r,s-1}$ as a vector field along α_{rs}, according to (A.9), (A.10). □

References

[1] Abraham R, Marsden J E and Ratiu T 1988 *Manifolds, Tensor Analysis, and Applications* (Berlin: Springer) 2nd edn

[2] Batlle C, Gomis J, Pons J M and Román-Roy N 1986 Equivalence between the Lagrangian and Hamiltonian formalism for constrained systems *J. Math. Phys.* **27** 2953–62

[3] Batlle C, Gomis J, Pons J M and Román-Roy N 1988 Lagrangian and Hamiltonian constraints for second-order singular Lagrangians *J. Phys. A: Math. Gen.* **21** 2693–703

[4] Bourbaki N 1983 *Varietés Différentielles et Analytiques* (Paris: Diffusion CCLS)

[5] Cariñena J F and López C 1992 The time evolution operator for higher-order singular Lagrangians *Int. J. Mod. Phys.* in press

[6] Cariñena J F, López C and Martínez E 1992 Sections along a map applied to higher-order Lagrangian mechanics. Noether's theorem *Acta Appl. Math.* in press

[7] Crampin M and Pirani F A E 1987 *Applicable Differential Geometry (London Math. Soc. Lecture Note Series 59)* (Cambridge: Cambridge University Press)

[8] Crampin M, Sarlet W and Cantrijn F 1986 Higher-order differential equations and higher-order Lagrangian mechanics *Math. Proc. Camb. Phil. Soc.* **99** 565–87

[9] Dieudonné J 1970 *Élements d'Analyse* vol 3 (Paris: Gauthier-Villars)

[10] Gotay M J and Nester J M 1980 Presymplectic Lagrangian systems II: the second-order equation problem *Ann. Inst. H Poincaré* A **32** 1–13

[11] Gotay M J, Nester J M and Hinds G 1978 Presymplectic manifolds and the Dirac-Bergmann theory of constraints *J. Math. Phys.* **19** 2388–99

[12] Gràcia X and Pons J M 1989 On an evolution operator connecting Lagrangian and Hamiltonian formalisms *Lett. Math. Phys.* **17** 175–80

[13] Gràcia X and Pons J M 1991 Constrained systems: a unified geometric approach *Int. J. Theor. Phys.* **30** 511–6

[14] Gràcia X and Pons J M 1992 A generalized geometric framework for constrained systems *Diff. Geom. Appl.* in press

[15] Gràcia X and Pons J M Lagrangian gauge transformations without Hamiltonian counterpart *Preprint* Universitat de Barcelona UB-ECM-PF 13/90

[16] Gràcia X, Pons J M and Román-Roy N 1991 Higher-order Lagrangian systems: geometric structuctures, dynamics, and constraints *J. Math. Phys.* **32** 2744–63

[17] de León M and Rodrigues P R 1985 *Generalized Classical Mechanics and Field Theory (North-Holland Mathematical Studies 112)* (Amsterdam: Elsevier)

[18] Muñoz M C and Román-Roy N 1992 Lagrangian theory for presymplectic systems *Ann. Inst. H Poincaré* in press

[19] Pons J M 1988 New relations between Hamiltonian and Lagrangian constraints *J. Phys. A: Math. Gen.* **21** 2705–15

[20] Saunders D J 1989 *The Geometry of Jet Bundles (London Math. Soc. Lecture Note Series 142)* (Cambridge: Cambridge University Press)

[21] Saunders D J and Crampin M 1990 On the Legendre map in higher-order field theories *J. Phys. A: Math. Gen.* **23** 3169–82

[22] Yano K and Ishihara S 1973 *Tangent and Cotangent Bundles: Differential Geometry (Pure and Applied Mathematics 16)* (New York: Marcel Dekker)